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MARTINGALES WITH VALUES IN UNIFORMLY CONVEX SPACES

BY

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ABSTRACT

Using the techniques of martingale inequalities in the case of Banach space valued martingales, we give a new proof of a theorem of Enflo: every super-reflexive space admits an equivalent uniformly convex norm. Let r be a number in $]2,\infty[$; we prove moreover that if a Banach space X is uniformly convex (resp. if $\delta_X(\varepsilon)/\varepsilon^r \to \infty$ when $\varepsilon \to 0$) then X admits for some $q < \infty$ (resp. for some $q < r$) an equivalent norm for which the corresponding modulus of convexity satisfies $\delta(\varepsilon)/\varepsilon^q \to \infty$ when $\varepsilon \to 0$. These results have dual analogues concerning the modulus of smoothness. Our method is to study some inequalities for martingales with values in super-reflexive or uniformly convex spaces which are characteristic of the geometry of these spaces up to isomorphism.

Introduction

In Section 1, we give (Th. 1.3) a martingale characterization of super-reflexive Banach spaces. On one hand it is related to Chatterji's results ([6]) on martingales with values in Banach spaces which have the Radon-Nikodym property, on the other hand it is related to a certain form of the strong law of large numbers for Banach space valued martingales. The latter was previously considered by A. Beck for martingales with independent increments in [3]. In Section 2, we prove similar inequalities for uniformly convex (or uniformly smooth) Banach spaces.

Section 3 contains our main results; there we show that the analogy between the inequalities of Sections 1 and 2 is not a mere coincidence: Theorem 3.1 gives a simple way to construct an equivalent uniformly convex (smooth) norm on a space X provided a certain inequality is satisfied by all X-valued martingales. The modulus of convexity δ obtained after such a renorming is "of power type": there are constants $K > 0$ and $q < \infty$ such that $\delta(\varepsilon) \geqq K\varepsilon^q$ for

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all $\varepsilon > 0$. Thus δ "dominates" the modulus of convexity of the spaces L_q. The modulus of convexity obtained by Enflo in [7] apparently does not have that property, it is only of "exponential type". In order to use Theorem 3.1 we have to show that super-reflexive spaces satisfy its hypothesis. This is done essentially in Lemma 3.1 which is the main technical difficulty of the paper. We use there different martingale inequality methods, especially stopping times techniques. Our results were announced in [23]. We refer to [21] for the probabilistic part and to [18] for Banach space theory.

Notations and conventions

To avoid referring repeatedly to a probability space, we shall call briefly martingale a sequence $(X_n)_{n\geqq 0}$ of Banach space valued integrable variables on some probability space $(\Omega', \mathcal{A}', \mathbf{P}')$ for which there exists an increasing sequence of sub-σ-algebras $(\mathcal{A}'_n)_{n\geqq 0}$ of $\mathcal{A}'$ such that

$$\mathbf{E}^{\mathcal{A}'_n}(X_{n+1}) = X_n \quad \text{for all} \quad n = 0, 1, 2, \cdots.$$

For every martingale $(X_n)_{n\geqq 0}$ we shall denote $(dX_n)_{n\geqq 1}$ the "increments" of the martingale $(X_n)_{n\geqq 0}$: $\forall n \geqq 1$: $dX_n = X_n - X_{n-1}$; moreover we always make the convention $dX_0 = X_0$.

$\mathbf{E}$ will denote the expectation on $(\Omega', \mathbf{P}')$. If $1 \leqq \alpha < \infty$ and Z is a Banach space valued random variable on $(\Omega', \mathbf{P}')$, we shall write simply $\| Z \|_\alpha$ for $(\int \| Z(\omega) \|^\alpha d\mathbf{P}'(\omega))^{1/\alpha}$; by $\| Z \|_\infty$ and $\| Z \|_{-\infty}$ we mean respectively

$$\operatorname*{ess\,sup}_{\omega \in \Omega'} \| Z(\omega) \| \quad \text{and} \quad \operatorname*{ess\,inf}_{\omega \in \Omega'} \| Z(\omega) \|.$$

Throughout this paper we reserve the notation $(\Omega, \mathcal{A}, \mathbf{P})$ for the space $\Omega = \{-1, +1\}^{\mathbf{N}}$ with its Borel σ-algebra $\mathcal{A}$ and the usual invariant probability $\mathbf{P}$. $\mathcal{A}_0$ will denote the trivial σ-algebra $\{\phi, \Omega\}$ on Ω and for all $n \geqq 1$ $\mathcal{A}_n$ will be the σ-algebra generated by the first n-coordinates on Ω denoted by $\varepsilon_1, \cdots, \varepsilon_n$. A martingale relative to $(\Omega, (\mathcal{A}_n)_{n\geqq 0}, \mathbf{P})$ will be called a Walsh-Paley martingale. This terminology is justified by the correspondence between such martingales and the partial sums of Walsh series for which Paley proved the first "martingale" inequalities. If X is a Banach space, its modulus of convexity δ_X is defined by:

$$\forall \varepsilon \in [0,2],\ \delta_X(\varepsilon) = \inf\left\{1 - \left\| \frac{x+y}{2} \right\| \quad \| x \| \leqq 1 \quad \| y \| \leqq 1 \quad \| x - y \| \geqq \varepsilon \right\}.$$

Its modulus of smoothness ρ_X is defined by

$$\forall t \in [0, \infty[,\ \rho_X(t) = \sup\left\{ \frac{\| x + ty \| + \| x - ty \|}{2} - 1 \quad \| x \| = \| y \| = 1 \right\}.$$

1. Preliminaries; martingales with values in super-reflexive spaces

Recall that the distance $d(E,F)$ between two Banach spaces E and F is $\inf \|T\| \ \|T^{-1}\|$, where T runs over all the isomorphisms from E onto F (with the convention $\inf \phi = +\infty$). We say (cf. [13]) that a Banach space E is finitely representable in a Banach space F if for every subspace M of E and every $\varepsilon > 0$ there is a subspace N of F such that $d(M,N) \leqq 1+\varepsilon$. Let $\mathcal{P}$ be a property concerning Banach spaces; we say that a Banach space E has the property super-$\mathcal{P}$ if all the Banach spaces which are finitely representable in E have the property $\mathcal{P}$. As immediate consequences of that definition, we have:

$$\text{super-}\mathcal{P} \Rightarrow \mathcal{P} \text{ and super-(super-}\mathcal{P}) \Leftrightarrow \text{super-}\mathcal{P}\,;$$

moreover, if Q is another property concerning Banach spaces, and if $\mathcal{P} \Rightarrow Q$, then super-$\mathcal{P} \Rightarrow$ super-Q.

Examples. 1) Let us consider for a Banach space X the following property: for some $\varepsilon > 0$ every subspace Y of X satisfies $d(Y, l_1) \geqq 1+\varepsilon$. It is easy to see that the associated super-property is: l_1 is not finitely representable in X.

2) The preceding example is rather simple; apparently the most interesting super-property is super-reflexivity, which has been introduced and studied by R. C. James in [13] and [14]. P. Enflo has obtained ([7]) the following fundamental result: every super-reflexive space has an equivalent uniformly convex norm. The converse had been previously proved by James in [13]. Several different properties of Banach spaces become, with the prefix super-, equivalent to super-reflexivity. (See for example: A. Brunel and L. Sucheston [4].)

Earlier results of James concerning weak compacity and reflexivity have analogues for super-reflexive spaces; for instance:

Theorem 1.1. (a) ([24]). *A Banach space X is super-reflexive if and only if there exists an integer n and an $\varepsilon > 0$ such that for every n-tuple $(x_1, \cdots, x_n)$ in the unit ball of X:*

$$\inf_{1 \leqq k \leqq n} \left\| \sum_{1 \leqq i \leqq k} x_i - \sum_{k < i \leqq n} x_i \right\| \leqq n(1-\varepsilon).$$

In that case, we say that X is $J-(n,\varepsilon)$ convex.

(b) ([13]). *A Banach space X is not super-reflexive if and only if for every ε in $]0,1[$ and every integer n there exists a subset $\{x_{\varepsilon_1,\cdots,\varepsilon_k} : 1 \leqq k \leqq n,\ \varepsilon_i = \pm 1\}$ of the unit ball of X with the properties that:*

(1.1) $$x_{\varepsilon_1,\cdots,\varepsilon_k} = \tfrac{1}{2}(x_{\varepsilon_1\cdots\varepsilon_k 1} + x_{\varepsilon_1\cdots\varepsilon_k -1})$$

and

(1.2) $$\| x_{\varepsilon_1\cdots\varepsilon_{k-1} 1} - x_{\varepsilon_1\cdots\varepsilon_{k-1} -1} \| \geqq 2\varepsilon$$

for every $k = 1,2,\cdots,n$ and every choice of signs (ε_i), $1 \leqq i \leqq n$. We then say that X has the finite tree property.

REMARK 1.1. The starting point of our work is to notice that the finite tree property can be translated in terms of martingales: in Theorem 1.1.b, let us define a sequence $(X_m)_{m\geqq 0}$ of random variables on $\{-1,+1\}^{\mathbf{N}}$ by:

$$X_0 = \tfrac{1}{2}(x_1 + x_{-1}) \qquad X_k(\varepsilon_1, \varepsilon_2, \cdots) = x_{\varepsilon_1\cdots\varepsilon_k}$$

for $k = 1,2,\cdots,n$, and $X_m = X_n$ for $m > n$.

The equality (1.1) means precisely that $(X_m)_{m\geqq 0}$ is a Walsh-Paley martingale, and (1.2) ensures that $\inf_{1\leqq k\leqq n} \| dX_k(\varepsilon_1,\varepsilon_2,\cdots)\| \geqq \varepsilon$ for every choice of signs $(\varepsilon_m)_{m\in\mathbf{N}}$.

This remark leads us to the following proposition, which is the link between our next results and the martingale characterization by Chatterji [6] of the Banach spaces which have the Radon-Nikodym property (in short the property RNP).

PROPOSITION 1.1. *Super-reflexivity is equivalent to the super-Radon-Nikodym property.*

PROOF. It is well known that reflexivity implies RNP, therefore super-reflexivity implies super-RNP. Conversely, if a Banach space E is not super-reflexive, then by Theorem 1.1.b (see [13] for more details), there exists a Banach space F finitely representable in E with the infinite tree property; that is to say: for some $\varepsilon > 0$ there exists a Walsh-Paley martingale $(X_m)_{m\geqq 0}$ with values in F and such that

$$\sup_n \| X_n \|_\infty \leqq 1 \quad \text{and} \quad \inf_{n\geqq 1} \inf_{(\varepsilon_i)\in\{-1,+1\}^{\mathbf{N}}} \| dX_n(\varepsilon_1,\varepsilon_2,\cdots)\| \geqq \varepsilon .$$

The latter property ensures that $(X_n)_{n\geqq 0}$ is everywhere non-convergent, therefore by Chatterji's Theorem ([6]) F does not have the RNP; as a consequence, E does not have the super-RNP, and this concludes the proof.

A sequence $(x_i)_{0\leqq i\leqq N}$ in a Banach space is called basic of constant b if for every sequence (α_i) of scalars and every $k \leqq N$

$$\left\| \sum_{i=0}^{k} \alpha_i x_i \right\| \leqq b \left\| \sum_{i=0}^{N} \alpha_i x_i \right\| .$$

A basic sequence of constant 1 is called a "monotone basic sequence".

We shall need the following result of James [14]:

THEOREM 1.2. *For each number b in* $]1,\infty[$, *a Banach space E is super-reflexive if and only if there exist a constant C and two numbers* $r > 1$ *and* $q < \infty$ *such that*

$$\frac{1}{C}\left(\sum \| x_n \|^q\right)^{1/q} \leqq \left\| \sum x_n \right\| \leqq C\left(\sum \| x_n \|^r\right)^{1/r}$$

for all finite sequences (x_n) *in E which are basic of constant b.*

REMARK 1.2. Let X be a Banach space and $(\Omega', \mathbf{P}')$ some probability space; it is clear that if $1 \leqq \alpha \leqq \infty$, and if $(X_n)_{n \geqq 0}$ is an X-valued martingale in $L_\alpha(\Omega', \mathbf{P}'; X)$, then the sequence $(dX_n)_{n \geqq 0}$ is a monotone basic sequence in $L_\alpha(\Omega', \mathbf{P}'; X)$. In order to apply the preceding theorem to that situation we need:

PROPOSITION 1.2. *Let* (Ω', μ) *be an arbitrary measure space with* $0 \neq \mu(\Omega') < \infty$ *and let* α *be such that* $1 < \alpha < \infty$. *A Banach space X is super-reflexive if and only if* $L_\alpha(\Omega', \mu; X)$ *is also super-reflexive.*

The above statement is well known; it is particularly clear if one knows Enflo's result in [7]. Since we wish to give a new proof of Enflo's theorem, we briefly indicate a direct proof.

PROOF. The "if" part is trivial; so assume that X is super-reflexive. By Theorem 1.1.a there exist n and $\varepsilon > 0$ such that X is J-(n, ε) convex. Since $1 < \alpha < \infty$, it is not difficult to show that there exists $\varepsilon' > 0$ such that

$$\inf_{1 \leqq k \leqq n} \left\| \sum_{1 \leqq i \leqq k} x_i - \sum_{k < i \leqq n} x_i \right\| \leqq (1 - \varepsilon') n \left(\frac{1}{n} \sum_{i=1}^{n} \| x_i \|^\alpha\right)^{1/\alpha}$$

for all (x_i) in X^n. Therefore:

$$\left(\frac{1}{n} \sum_{1 \leqq k \leqq n} \left\| \sum_{1 \leqq i \leqq k} x_i - \sum_{k < i \leqq n} x_i \right\|^\alpha\right)^{1/\alpha} \leqq \left(\frac{n - 1 + (1 - \varepsilon')^\alpha}{n}\right)^{1/\alpha} \cdot n \left(\frac{1}{n} \sum_{i=1}^{n} \| x_i \|^\alpha\right)^{1/\alpha}.$$

It is clear that the last inequality remains valid if $(x_1, \cdots, x_n)$ is an n-tuple of elements of $L_\alpha(\Omega', \mu; X)$. A fortiori, we obtain:

$$\inf_{1 \leqq k \leqq n} \left\| \sum_{1 \leqq i \leqq k} x_i - \sum_{k < i \leqq n} x_i \right\| \leqq \left(\frac{n - 1 + (1 - \varepsilon')^\alpha}{n}\right)^{1/\alpha} n \sup \| x_i \|$$

for every n-tuple $(x_1, \cdots, x_n)$ in $L_\alpha(\Omega', \mu; X)$. Since

$$\left(\frac{n-1+(1-\varepsilon')^{\alpha}}{n}\right)^{1/\alpha}<1,$$

we conclude (Th. 1.1.a) that $L_\alpha(\Omega',\mu;X)$ is super-reflexive.

The origin of the next theorem is a remark of S. Kwapień.

THEOREM 1.3. *Let X be a Banach space; the following properties are equivalent:*

(i) *X is super-reflexive.*

(ii) *For every α in $]1,\infty[$, there exist a constant C and $r>1$ such that all X-valued martingales $(X_n)_{n\geqq 0}$ satisfy:*

$$\sup_n \|X_n\|_\alpha \leqq C\Big(\sum_{n\geqq 0}\|dX_n\|_\alpha^r\Big)^{1/r}. \tag{1.3}$$

(iii) *There exist a constant C and $r>1$ such that all X-valued Walsh-Paley martingales $(X_n)_{n\geqq 0}$ satisfy:*

$$\sup_n \|X_n\|_{-\infty} \leqq C\Big(\sum_{n\geqq 0}\|dX_n\|_\infty^r\Big)^{1/r}. \tag{1.4}$$

(iv) *For every α in $]1,\infty[$ there exist a constant C and $q<\infty$ such that all X-valued martingales $(X_n)_{n\geqq 0}$ satisfy:*

$$\Big(\sum_{n\geqq 0}\|dX_n\|_\alpha^q\Big)^{1/q} \leqq C\sup_n \|X_n\|_\alpha. \tag{1.5}$$

(v) *There exist a constant C and $q<\infty$ such that all X-valued Walsh-Paley martingales $(X_n)_{n\geqq 0}$ satisfy:*

$$\Big(\sum_{n\geqq 0}\|dX_n\|_{-\infty}^q\Big)^{1/q} \leqq C\sup \|X_n\|_\infty. \tag{1.6}$$

PROOF. (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iv) follow from Remark 1.2, Proposition 1.2 and Theorem 1.2 applied to the Banach spaces $E=L_\alpha(\Omega',\mathbf{P}';X)$, $(\Omega',\mathbf{P}')$ being an arbitrary probability space. The fact that the constants involved (namely C, r and q) do not depend on the probability space $(\Omega',\mathbf{P}')$ follows from the obvious remark that the space $L_\alpha(\Omega',\mathbf{P}';X)$ is finitely representable in $l_\alpha(X)$. (ii) $\Rightarrow$ (iii) and (iv) $\Rightarrow$ (v) are both trivial.

(v) $\Rightarrow$ (i): If X is not super-reflexive, then (Theorem 1.1.b) X has the finite tree property; that is to say (see Remark 1.1), for every integer n there exists an X-valued Walsh-Paley martingale $(X_m)_{m\geqq 0}$ with the properties that:

$$\|X_n\|_\infty \leqq 1 \quad\text{and}\quad \inf_{1\leqq k\leqq n}\|dX_k\|_{-\infty}\geqq 1/2.$$

Assuming that (1.6) is satisfied, we get:

$$n^{1/q} \cdot (1/2) \leqq C,$$

which is a contradiction when n is large enough. The only remaining implication is (iii) $\Rightarrow$ (i): we assume that X is not super-reflexive; since (cf. [13]) super-reflexivity is a self-dual property, X^* is also non super-reflexive. Therefore, by Theorem1.1.b, for each integer n and each $\varepsilon < 1$, there exists an X^*-valued martingale $(X'_m)_{m \geqq 0}$ such that $\|X'_n\|_\infty \leqq 1$ and $\inf_{1 \leqq k \leqq n} \|dX'_k\|_{-\infty} \geqq \varepsilon$. Since $\varepsilon_k \cdot dX'_k$ is $\mathcal{A}_{k-1}$-measurable, for every $\theta > 0$ there exists an $\mathcal{A}_{k-1}$-measurable variable Y_{k-1} with values in the unit ball of X such that $\langle Y_{k-1}(\omega), \varepsilon_k dX'_k(\omega)\rangle \geqq \varepsilon - \theta$ for every ω in Ω, and every $k = 1,2,\cdots,n$. If we form the X-valued Walsh-Paley martingale (uniquely) defined by $X_m = \Sigma_{k=1}^n \varepsilon_k Y_{k-1}$ for $m \geqq n$, we can observe that:

$$n(\varepsilon - \theta) \leqq \sum_{k=1}^{n} \mathbf{E}\langle dX_k, dX'_k\rangle = \mathbf{E}\langle X_n, X'_n\rangle \leqq \mathbf{E}\|X_n\|.$$

But on the other hand, $\|X_n(\omega)\| \leqq \Sigma_{k=1}^n \|Y_{k-1}(\omega)\| \leqq n$ for all ω in Ω; therefore a standard computation shows that $\|X_n\|_{-\infty} \geqq \varphi_n(\varepsilon,\theta)n$ for some function $\varphi_n(\varepsilon,\theta)$ such that $\varphi_n(\varepsilon,\theta) \to 1$ when $\varepsilon \to 1$ and $\theta \to 0$. Hence we can choose ε and θ such that $\|X_n\|_{-\infty} \geqq n/2$. If we now assume that (1.4) is satisfied, we get $n/2 \leqq Cn^{1/r}$, which is imposible as soon as n is large enough. A contradiction that concludes the proof.

REMARK 1.3. The above properties are also equivalent to:

(vi) Every X-valued Walsh-Paley martingale $(X_m)_{m \geqq 0}$ such that $\sup_n \|dX_n\|_\infty < \infty$ has the property that X_n/n tends to zero almost surely.

Assume (ii) and let $(X_n)_{n \geqq 0}$ be as in (vi); it follows from (1.3) that the martingale $(S_n)_{n \geqq 0}$ defined by $S_n = \Sigma_{m=1}^n dX_m/m$ converges to a variable S_∞ in $L_\alpha(X)$ when $n \to \infty$. By the martingale convergence theorem, S_n converges almost surely to S_∞ when $n \to \infty$. By Kronecker's Lemma ([21], p. 151) almost surely S_n/n tends to zero when $n \to \infty$. This proves that (ii) $\Rightarrow$ (vi). We only sketch the proof of (vi) $\Rightarrow$ (i): if X is not super-reflexive, then, by the argument used for (iii) $\Rightarrow$ (i), for every $\delta < 1$ and every integer n there exists an X-valued Walsh-Paley martingale $(X_m)_{m \geqq 0}$ such that $\|X_n\|_{-\infty} \geqq \delta n$ and $\|dX_k\|_\infty \leqq 1$ for $k = 1,2,\cdots,n$. An idea due to A. Beck (see [20], Exp. VII, Prop. 7) allows to construct an infinite Walsh-Paley martingale $(Z_m)_{m \geqq 0}$ such that $\sup_n \|dZ_n\|_\infty \leqq 1$ but also $\forall \omega \in \Omega$, $\limsup_{n \to \infty} \|Z_n(\omega)\|/n > 0$; this contradicts (vi) and concludes the proof.

REMARK 1.4. In Theorem 1.3, considering martingales is really a necessity: if one replaces in properties (ii), (iii) and (vi) the word "martingale" by "martingale with independent increments", then one obtains properties all equivalent to "l_1 is not finitely representable in X" (cf. for example [20], Exp. VII); such a property is strictly weaker than super-reflexivity: because of a remarkable counterexample of R. C. James [15], there exists a non-reflexive Banach space in which l_1 is not finitely representable (more precisely: in which even $l_1^{(3)}$ is not finitely representable). If one replaces again in properties (iv) and (v) "martingale" by "martingale with independent increments", then one obtains properties equivalent to $\ll l_\infty$ is not finitely representable in $X \gg$ (see [19] §1). It is clear that the latter property is strictly weaker than super-reflexivity, since l_1 has it.

2. Martingales with values in uniformly convex spaces

In a preliminary version of this work, I proved the inequalities of Section 2 for Walsh-Paley martingales; P. Assouad pointed out to me that it was possible to extend them; he extended them further in [1] to obtain several results on rearrangements of series in uniformly smooth spaces. T. Figiel has observed that his results (as used in Proposition 2.1) give a nicer form to these inequalities. The results concerning monotone basic sequences seem to go back to [26].

THEOREM 2.1 (T. Figiel [9]). (a) *There exist constants R and $c > 0$ such that for any Banach space X of dimension at least* 2:

$$\forall \varepsilon \in \ ]0,2], \ \frac{1}{R}\delta_X(c\varepsilon) \leqq \delta_{L^2(X)}(\varepsilon) \leqq \delta_X(\varepsilon)$$

$$\forall t \in \ ]0,\infty[, \ \rho_X(t) \leqq \rho_{L^2(X)}(t) \leqq R\rho_X(t/c),$$

where $L^2(X)$ denotes the space $L^2(\Omega,\mu;X)$ for an arbitrary non trivial measure space (Ω,μ).

(b) *For any Banach space X, the function $\varepsilon \to \delta_X(\varepsilon)/\varepsilon$ is increasing on* $]0,2]$.

PROPOSITION 2.1. *If $(x_n)_{n\geqq 0}$ is a monotone basic sequence in a Banach space X satisfying*

$$\sup_n \left\| \sum_{i=0}^{i=n} x_i \right\| \leqq 1$$

then

$$\| x_0 \| + \sum_{n=1}^{\infty} \delta_X(\| x_n \|) \leqq 1.$$

PROOF. Let n be an integer, we write S_n for $\Sigma_{i=0}^{i=n} x_i$; the definition of δ_X gives: (since $\| S_n \| \leqq \| S_{n+1} \|$),

$$\frac{1}{2}\left\| \frac{S_{n+1}}{\| S_{n+1} \|} + \frac{S_n}{\| S_{n+1} \|} \right\| \leqq 1 - \delta_X\left(\frac{\| x_{n+1} \|}{\| S_{n+1} \|}\right),$$

or:

$$\left\| S_n + \frac{x_{n+1}}{2} \right\| + \| S_{n+1} \| \, \delta_X\left(\frac{\| x_{n+1} \|}{\| S_{n+1} \|}\right) \leqq \| S_{n+1} \|. \tag{2.1}$$

If $\| S_{n+1} \| \leqq 1$, Theorem 2.1.b implies that

$$\delta_X(\| x_{n+1} \|) \leqq \| S_{n+1} \| \, \delta_X\left(\frac{\| x_{n+1} \|}{\| S_{n+1} \|}\right);$$

moreover, by the monotony of the sequence $(x_n)_{n \geqq 0}$:

$$\| S_n \| \leqq \left\| S_n + \frac{x_{n+1}}{2} \right\|.$$

Therefore, (2.1) gives as a consequence:

$$\| S_n \| + \delta_X(\| x_{n+1} \|) \leqq \| S_{n+1} \|;$$

adding these inequalities when $n \geqq 0$ we get:

$$\| x_0 \| + \sum_{n \geqq 0} \delta_X(\| x_{n+1} \|) \leqq \sup_{n \geqq 0} \| S_{n+1} \| \leqq 1,$$

which concludes the proof.

The above proposition has the following dual analogue:

PROPOSITION 2.2. *If $(x_n)_{n \geqq 0}$ is a monotone basic sequence in a Banach space X satisfying*

$$\sum_{n \geqq 0} \rho_X(\| x_n \|) \leqq 1$$

then:

$$\sup_n \left\| \sum_{i=0}^{n} x_i \right\| \leqq 4.$$

The proof follows indeed from Proposition 2.1 and Lindenstrauss' duality formulae:

$$\forall t > 0, \quad \rho_X(t) = \sup_{0<\varepsilon \leqq 2} \left[\frac{t\varepsilon}{2} - \delta_{X^*}(\varepsilon) \right]$$

$$\forall \varepsilon \in\ ]0,2], \quad \delta_{X^*}(\varepsilon) \geqq \sup_{t>0} \left[\frac{t\varepsilon}{2} - \rho_X(t) \right]. \quad \text{(See [17].)}$$

In the sequel we make the convention that $\delta_X(\varepsilon) = +\infty$ whenever X is a Banach space and $\varepsilon > 2$.

If φ is an increasing function from $\mathbf{R}_+$ to $\bar{\mathbf{R}}_+$ such that $\varphi(0) = 0$, we shall define the Orlicz gauge $\|\cdot\|_\varphi$ on $\mathbf{R}^\mathbf{N}$ in the usual way:

$$\forall (\alpha_n) \in \mathbf{R}^\mathbf{N}, \ \|(\alpha_n)_n\|_\varphi = \inf \left\{ c > 0 \,\middle|\, \sum_{n \in N} \varphi\left(\frac{|\alpha_n|}{c}\right) \leqq \varphi(1) \right\}.$$

Using Theorem 2.1, the above Propositions and Remark 1.2, we get:

THEOREM 2.2. *There are constants K, L such that for any Banach space X of dimension at least* 2 *and any X-valued square-integrable martingale* $(X_n)_{n \geq 0}$ *the following inequalities hold*:

(2.2)
$$\frac{1}{K} \|(\| dX_n \|_2)_{n \geqq 0}\|_{\delta_X} \leqq \sup_{n \geqq 0} \| X_n \|_2 \leqq L \, \|(\| dX_n \|_2)_{n \geqq 0}\|_{\rho_X}.$$

REMARK 2.1. i) When $\dim X = 1$, (2.2) is valid if $K = L = 1$ and if we substitute to both ρ_X and δ_X the function $t \to t^2$. Obviously nothing stronger can be said.

ii) The inequalities appearing in Theorem 2.2 can be considered as a generalization respectively of a result of Kadec and of Lindenstrauss ([18], Th. II.3.6). It should be observed that it is enough to prove only one of the inequalities in (2.2), the other can then be proved using Lindenstrauss' duality formula. It is also of interest to notice that if the Walsh-Paley X-valued martingales satisfy the left part (resp. the right part) of (2.2) for some constant K (resp. L) and *some* function δ_X strictly positive on $]0,2]$ (resp. ρ_X such that $\rho_X(t)/t \to 0$ when $t \to 0$), then the space X is super-reflexive since it does not have the finite tree property (resp. since its dual does not have the finite tree property).

In [11], Theorem 2.2 is proved for martingales which are the partial sums of a Rademacher series with coefficients in X.

In the preceding Theorem we have considered square integrable martingales since Hilbert spaces are known to be the "most" uniformly convex and the "most" uniformly smooth among Banach spaces of dimension at least 2 (cf. [22]). For other cases we shall need the following result of T. Figiel ([8], Prop. 1 and added in proof):

PROPOSITION 2.3. *Let q and p be respectively in $[2,\infty[$ and in $]1,2]$. If a Banach space X satisfies $\forall\varepsilon \in]0,2]$, $\delta_X(\varepsilon) \geqq A\varepsilon^q$ (resp. $\forall t \in]0,\infty[$ $\rho_X(t) \leqq At^p$) for some constant $A>0$, then there exists a constant $B>0$ (depending only on A and q or p) such that $\forall\varepsilon \in]0,2]$, $\delta_{L_q(\Omega',\mu;X)}(\varepsilon) \geqq B\varepsilon^q$ (resp. $\forall t \in]0,\infty[$, $\rho_{L_p(\Omega',\mu;X)}(t) \leqq Bt^p$) for any measure space (Ω',μ).*

PROPOSITION 2.4. *Let X be a Banach space.*

(a) *If for all ε in $]0,2]$, $\delta_X(\varepsilon) \geqq A\varepsilon^q$ for some constant $A>0$ and some q in $[2,\infty[$, than there exists a constant $B>0$ such that*

$$\mathbf{E}\|X_0\|^q + B\sum_{n\geqq 1}\mathbf{E}\|dX_n\|^q \leqq \sup_{n\geqq 0}\mathbf{E}\|X_n\|^q$$

for all X-valued martingales $(X_n)_{n\leqq 0}$.

(b) *If for all t in $]0,\infty[$, $\rho_X(t) \leqq At^p$ for some constant A and some p in $]1,2]$, then there exists a constant Δ such that*

$$\sup_n \mathbf{E}\|X_n\|^p \leqq \mathbf{E}\|X_o\|^p + \Delta\sum_{n\geqq 1}\mathbf{E}\|dX_n\|^p$$

for all X-valued martingales $(X_n)_{n\geqq 0}$.

PROOF. (b) follows from (a) by an easy argument of duality, so we prove only (a). By Proposition 2.3 we may assume that if $(\Omega',\mathbf{P}')$ is any probability space: $\forall\varepsilon \in]0,2]$, $\delta_{L_q(\Omega',\mathbf{P}';X)}(\varepsilon) \geqq B\varepsilon^q$ for a constant $B>0$. If $(X_n)_{n\geqq 0}$ is an X-valued martingale in $L_q(\Omega',\mathbf{P}';X)$, then $(dX_n)_{n\geqq 0}$ is a monotone basic sequence in $L_q(\Omega',\mathbf{P}';X)$; applying Proposition 2.1 we get that $\sup_{n\geqq 0}\mathbf{E}\|X_n\|^q \leqq 1$ implies

$$(\mathbf{E}\|X_0\|^q)^{1/q} + B\sum_{n\geqq 1}\mathbf{E}\|dX_n\|^q \leqq 1,$$

hence $\mathbf{E}\|X_0\|^q + B\sum_{n\geqq 1}\mathbf{E}\|dX_n\|^q \leqq 1$, which concludes the proof by homogeneity.

3. Equivalent norms

In order to shorten the terminology we shall say that a Banach space X is p-smooth, $1<p\leqq 2$, if there exists an equivalent norm on X for which the modulus of smoothness ρ satisfies $\forall t>0$, $\rho(t)\leqq Ct^p$ for some constant C. We shall say that X is q-convex, $2\leqq q<\infty$, if there exists an equivalent norm on X for which the modulus of convexity δ satisfies $\forall\varepsilon>0$, $\delta(\varepsilon)\geqq C\varepsilon^q$ for some constant $C>0$.

Assume that a Banach space $(X, \| \ \|)$ is q-convex (resp. p-smooth), then by Proposition 2.4 there exists a constant C_1 such that all X-valued martingales $(X_n)_{n \geqq 0}$ satisfy:

$$\mathbf{E}\| X_0 \|^q + \sum_{n \geqq 1} \mathbf{E}\| dX_n \|^q \leqq C_1 \sup_n \mathbf{E}\| X_n \|^q$$

$$\left(\text{resp.} \sup \mathbf{E}\| X_n \|^p \leqq C_1 \left[\mathbf{E}\| X_0 \|^p + \sum_{n \geqq 1} \mathbf{E}\| dX_n \|^p\right]\right).$$

The following Theorem is a converse to the preceding remark:

THEOREM 3.1. *Let p and q be such that $1 \leqq p, q < \infty$ and let X be a Banach space.*

(a) *Assume that there exists a constant C for which all X-valued Walsh-Paley martingales $(X_n)_{n \geqq 0}$ satisfy:*

$$\mathbf{E}\| X_o \|^q + \sum_{n \geqq 1} \mathbf{E}\| dX_n \|^q \leqq C^q \sup_n \mathbf{E}\| X_n \|^q, \tag{3.1}$$

then there exists an equivalent norm $| \ |$ on X such that:

$$\begin{cases} \forall x, y \in X, \ \| x \| \leqq | x | \leqq C \| x \| \\ \text{and } \left| \dfrac{x+y}{2} \right|^q \leqq \dfrac{| x |^q + | y |^q}{2} - \left\| \dfrac{x-y}{2} \right\|^q ; \end{cases} \tag{3.2}$$

in particular, the modulus of convexity δ of $(X, | \ |)$ satisfies:

$$\forall \varepsilon > 0, \delta(\varepsilon) \geqq (1/q)(\varepsilon/2C)^q.$$

(b) *Assume that there exists a constant C for which all X-valued Walsh-Paley martingales $(X_n)_{n \geqq 0}$ satisfy:*

$$\sup_n \mathbf{E}\| X_n \|^p \leqq C^p \left(\mathbf{E}\| X_0 \|^p + \sum_{n \geqq 1} \mathbf{E}\| dX_n \|^p \right), \tag{3.3}$$

then there exists an equivalent norm $| \ |$ on X such that:

$$\begin{cases} \forall x, y \in X, \ \dfrac{1}{C}\| x \| \leqq | x | \leqq \| x \| \\ \text{and } \dfrac{| x+y |^p + | x-y |^p}{2} \leqq | x |^p + \| y \|^p ; \end{cases} \tag{3.4}$$

in particular, the modulus of smoothness ρ of $(X, | \ |)$ satisfies $\forall t > 0$, $\rho(t) \leqq C^p t^p$.

PROOF. (a) For all x in X we define $|x|$ as

$$\inf\left\{\left(C^q \sup_n \mathbf{E}\|X_n\|^q - \sum_{n\geq 1}\mathbf{E}\|dX_n\|^q\right)^{1/q}\right\},$$

where the infimum runs over all X-valued Walsh-Paley martingales $(X_n)_{n\geq 0}$ such that $X_0 = x$ and $\sup_n \mathbf{E}\|X_n\|^q < \infty$. (3.1) gives us $\|x\| \leq |x|$; on the other hand, if we consider the Walsh-Paley martingale $(X_n)_{n\geq 0}$ defined by $\forall n \geq 0$, $X_n \equiv x$, we get $|x| \leq C\|x\|$. Let x and y be elements of X, by the definition of $|x|$ and $|y|$, for all $\gamma > 0$ there exist Walsh-Paley martingales $(X_n)_{n\geq 0}$ and $(Y_n)_{n\geq 0}$ such that:

$$X_0 = x,\ \sup \mathbf{E}\|X_n\|^q < \infty$$

$$Y_0 = y,\ \sup \mathbf{E}\|Y_n\|^q < \infty \quad \text{and}$$

$$C^q \sup \mathbf{E}\|X_n\|^q - \sum_{n\geq 1}\mathbf{E}\|dX_n\|^q \leq |x|^q + \gamma$$

$$C^q \sup \mathbf{E}\|Y_n\|^q - \sum_{n\geq 1}\mathbf{E}\|dY_n\|^q \leq |y|^q + \gamma.$$

We then build a new Walsh-Paley martingale $(Z_n)_{n\geq 0}$ by setting: $Z_0 = (x+y)/2$ and

$$\forall n \geq 1,\ Z_n(\varepsilon_1, \varepsilon_2, \cdots) = \left(\frac{1+\varepsilon_1}{2}\right)X_{n-1}(\varepsilon_2, \varepsilon_3, \cdots) + \left(\frac{1-\varepsilon_1}{2}\right)Y_{n-1}(\varepsilon_2, \varepsilon_3, \cdots);$$

since $\sup_n \mathbf{E}\|Z_n\|^q = \sup_n \frac{1}{2}(\mathbf{E}\|X_n\|^q + \mathbf{E}\|Y_n\|^q) < \infty$, we can write:

$$\begin{aligned}\left|\frac{x+y}{2}\right|^q &\leq C^q \sup_n \mathbf{E}\|Z_n\|^q - \sum_{n\geq 1}\mathbf{E}\|dZ_n\|^q \\ &\leq C^q \sup_n \frac{\mathbf{E}\|X_n\|^q + \mathbf{E}\|Y_n\|^q}{2} - \left\|\frac{x-y}{2}\right\|^q - \sum_{n\geq 1}\frac{\mathbf{E}\|dX_n\|^q + \mathbf{E}\|dY_n\|^q}{2} \\ &\leq \frac{|x|^q + |y|^q}{2} + \gamma - \left\|\frac{x-y}{2}\right\|^q.\end{aligned}$$

Since $\gamma > 0$ is arbitrary, we obtain the announced result. The function $x \to |x|$ is obviously homogeneous on X and from the inequality we just proved follows that $\{x \in X \mid |x| \leq 1\}$ is convex; hence $x \to |x|$ is a norm on X. Now if $|x| \leq 1$, $|y| \leq 1$ and $|x-y| \geq \varepsilon$, we have

$$\left|\frac{x+y}{2}\right| \leq \left(1 - \left\|\frac{x-y}{2}\right\|^q\right)^{1/q} \leq \left(1 - \left(\frac{\varepsilon}{2C}\right)^q\right)^{1/q} \leq 1 - \frac{1}{q}\left(\frac{\varepsilon}{2C}\right)^q,$$

which settles the last assertion.

(b) To prove this part two ways are possible:

(1) One is to define first $\{x\}$ as

$$\sup\left\{\left(\frac{1}{C^p}\sup_n \mathbf{E}\|X_n\|^p - \sum_{n\geqq 1}\mathbf{E}\|dX_n\|^p\right)^{1/p}\right\},$$

where the supremum runs over all X-valued Walsh-Paley martingales such that $X_0 = x$ and $\sup_n \mathbf{E}\|X_n\|^p < \infty$; and then to define $|x|$ as $\inf\{\Sigma_{i\in I}\{x_i\}\}$, where the infimum is on all the finite subsets $(x_i)_{i\in I}$ in X such that $\Sigma_{i\in I}x_i = x$. The same idea as in the proof of (a) shows that $|\ |$ has the desired properties.

(2) Another way is to use duality: If all X-valued Walsh-Paley martingales $(X_n)_{n\geqq 0}$ satisfy (3.3), then all X^*-valued Walsh-Paley martingales $(X_n)_{n\geqq 0}$ satisfy (3.1) with $1/p + 1/q = 1$; appling part (a) to the space X^* we obtain an equivalent norm $|\ |^*$ on X^* and it is easy to check that the norm $|\ |$ on X which is the dual of the norm $|\ |^*$ on X^* satisfies the conclusions in (b) if and only if $|\ |^*$ satisfies the conclusions in (a).

REMARK 3.1. The precise converse to Theorem 3.1 is evidently true: If $|\ |$ and $\|\ \|$ are two norms on X satisfying (3.2) [resp. (3.4)], then all X-valued Walsh-Paley martingales satisfy (3.1) [resp. (3.3)].

A particular case of Theorem 3.1 is:

COROLLARY 3.1. *Let $(h_n)_{n\in\mathbf{N}}$ be the Haar orthonormal system on the Lebesgue interval. A Banach space X is 2-convex (resp. 2-smooth) if and only if there exists a constant C such that*

$$\left(\sum\|x_n\|^2\right)^{1/2} \leqq C\left(\int\left\|\sum h_n(t)x_n\right\|^2 dt\right)^{1/2}$$

$$\left[\textit{resp.}\ (3.5)\quad \left(\int\left\|\sum h_n(t)x_n\right\|^2 dt\right)^{1/2} \leqq C\left(\sum\|x_n\|^2\right)^{1/2}\right]$$

for all sequences $(x_n)_{n\in\mathbf{N}}$ of elements of X among which only a finite number are not zero.

REMARK 3.2. If X is 2-convex and 2-smooth, then Corollary 3.1 and the Theorem 1 in [16] imply that X is isomorphic to a Hilbert space. This result was previously proved in [11].

Corollary 3.1 gives a complete answer to a problem of [16] (conjecture and Remark 1). S. Kwapień noticed shortly after [16] appeared that there exist spaces (namely L^p for $p > 2$) which satisfy (3.5) but are not isomorphic to a Hilbert space.

The following result improves Enflo's Theorem [7]. In the case of spaces with local unconditional structure, it is due to Figiel and Johnson [10].

THEOREM 3.2. *Every super-reflexive space is q-convex and p-smooth for some $q < \infty$ and some $p > 1$.*

The main difficulty here is the following Lemma, the proof of which we postpone till the end of this paragraph.

LEMMA 3.1. *Let r be a number in* $]1,2]$ *and let X be a Banach space; assume that — for some constant D — all the X-valued martingales* $(X_m)_{m \geqq 0}$ *satisfy:*

$$(3.6)\ \forall n \in \mathbf{N} \qquad \| X_n \|_2 \leqq D(n+1)^{1/r} \sup_{0 \leqq k \leqq n} \| dX_k \|_\infty ;$$

then for all $p < r$ *there exists a constant* C_p *for which all X-valued Walsh-Paley martingales* $(X_m)_{m \geqq 0}$ *satisfy:*

$$\sup_n \mathbf{E} \| X_n \|^p \leqq C_p \left(\mathbf{E} \| X_0 \|^p + \sum_{n \geqq 1} \mathbf{E} \| dX_n \|^p \right).$$

Therefore, by Theorem 3.1, *X is p-smooth.*

PROOF OF THEOREM 3.2. It is an easy consequence of Lindenstrauss' duality formula that if X^* is p-smooth, then X is q-convex with $1/q + 1/p = 1$. If X is super-reflexive, then (cf. [13]) X^* is also super-reflexive, therefore it is enough to prove that a super-reflexive space is p-smooth for some $p > 1$. Now if X is super-reflexive, Theorem 1.3 ensures that the assumption of Lemma 3.1 is satisfied for some $r > 1$; hence the conclusion of Lemma 3.1 is valid and if $1 < p < r$, the space X is p-smooth.

The new result in Theorem 3.2 is that every uniformly convex or uniformly smooth space is q-convex and p-smooth for some $q < \infty$ and $p > 1$ (such spaces are super-reflexive by the results of [13]). This can be extended further to:

THEOREM 3.3.

(a) *If a Banach space X satisfies* $\rho_X(t)/t^\alpha \to 0$ *when* $t \to 0$ *for some* α *in* $[1,2[$, *then there exists a number* $p > \alpha$ *for which X is p-smooth.*

(b) *If a Banach space X satisfies* $\delta_X(\varepsilon)/\varepsilon^\alpha \to \infty$ *when* $\varepsilon \to 0$ *for some* α *in* $]2,\infty[$, *then there exists a number* $q < \alpha$ *for which X is q-convex.*

As above, by an easy duality argument it is enough to prove only (a). We shall need some extra notation: we define for each integer $n \geqq 1$ the number γ_n^X as the smallest positive constant γ for which all X-valued martingales $(X_m)_{m \geqq 0}$ such that $X_0 = 0$ satisfy

$$\| X_n \|_2 \leqq \gamma \sup_{1 \leqq k \leqq n} \| dX_k \|_2 .$$

Obviously: $\forall n \geqq 1$, $\gamma_n^X \leqq n$. The introduction of the numbers γ_n^X is motivated by:

LEMMA 3.2. $\forall n, k \geqq 1$, $\gamma_{nk}^X \leqq \gamma_n^X \gamma_k^X$.

PROOF OF LEMMA 3.2. If $(X_m)_{m \geqq 0}$ is an X-valued martingale such that $X_0 = 0$, then $(X_{mk})_{m \geqq 0}$ has the same property; hence

$$\| X_{nk} \|_2 \leqq \gamma_n^X \sup_{1 \leqq m \leqq n} \| X_{mk} - X_{(m-1)k} \|_2 . \tag{3.7}$$

Now set $Z_j^m = X_{(m-1)k+j} - X_{(m-1)k}$; $(Z_j^m)_{j \geqq 0}$ is (for each $m \geqq 1$) a martingale such that $Z_0^m = 0$, therefore:

$$\forall m = 1,2,\cdots \quad \| Z_k^m \|_2 \leqq \gamma_k^X \sup_{1 \leqq j \leqq k} \| dZ_j^m \|_2 ,$$

or equivalently:

$$\forall m = 1,2,\cdots \quad \| X_{mk} - X_{(m-1)k} \|_2 \leqq \gamma_k^X \sup_{(m-1)k < j \leqq mk} \| dX_j \|_2 .$$

Combining this last inequality with (3.7) we get

$$\| X_{nk} \|_2 \leqq \gamma_n^X \gamma_k^X \sup_{1 \leqq j \leqq nk} \| dX_j \|_2 ,$$

which proves that $\gamma_{nk}^X \leqq \gamma_n^X \gamma_k^X$.

PROOF OF THEOREM 3.3.a. If $\rho_X(t)/t^\alpha \to 0$ when $t \to 0$, an easy computation shows that $\rho_n / n^{1/\alpha} \to 0$ when $n \to \infty$, where we have denoted by ρ_n the number $\|(1,1,\cdots,1,0,0,\cdots)\|_{\rho_X}$, where 1 is repeated n times.

By Theorem 2.2 (excluding the trivial case $X = \mathbf{R}$), we know that all X-valued martingales such that $X_0 = 0$ must satisfy for all $n \geqq 1$

$$\| X_n \|_2 \leqq L \rho_n \sup_{1 \leqq m \leqq n} \| dX_m \|_2 ;$$

in other words, we have that for all $n \geqq 1$, $\gamma_n^X \leqq L\rho_n$, therefore we also have that $\gamma_n^X / n^{1/\alpha} \to 0$ when $n \to \infty$. But it is an easy consequence of Lemma 3.2 and the fact that $n \to \gamma_n^X$ is increasing that $\gamma_n^X / n^{1/\alpha} \to$ when $n \to \infty$ only if there exist a number $r > \alpha$ and a constant D such that $\gamma_n^X \leqq D n^{1/r}$ for all $n \geqq 1$. We are in a position to apply Lemma 3.1 and we obtain that X is p-smooth as soon as $\alpha < p < r$, which concludes the proof.

One should notice that we obtain the conclusion as soon as there exists an integer N for which $\gamma_N^X < N^{1/\alpha}$, which is apparently a stronger result.

PROOF OF LEMMA 3.1. We split the proof into several sublemmas:

SUBLEMMA 3.1. *Under the assumption of Lemma* 3.1, *for all $p < r$ there exists a constant α_p such that all X-valued Walsh-Paley martingales $(X_m)_{m\geqq 0}$ satisfy*:

$$\sup_n \| X_n \|_2 \leqq \alpha_p \left\| \left(\sum_{n\geqq 0} \| dX_n \|^p \right)^{1/p} \right\|_\infty . \tag{3.8}$$

PROOF. Let $(X_m)_{m\geqq 0}$ be an X-valued Walsh-Paley martingale; to prove (3.8), there is no loss of generality in assuming that $(X_m)_{m\geqq 0}$ is finite, i.e. that $dX_m = 0$ if m is greater than some integer N. We have then: $\sup_n \| X_n \|_2 = \| X_N \|_2$. Recall that $\mathcal{A}_n$ is the σ-algebra on $\Omega = \{-1, +1\}^{\mathbf{N}}$ generated by the n first coordinates; we shall make use of the fact that for all $n \geqq 1$ the variable $\omega \to \| dX_n(\omega) \|$ is $\mathcal{A}_{n-1}$-measurable. For simplicity, we set

$$S = \left\| \left(\sum_{n\geqq 0} \| dX_n \|^p \right)^{1/p} \right\|_\infty$$

and proceed to prove (3.8).
For each integer $k \geqq 0$, we define a random subset $A_k(\omega)$ of $\mathbf{N}$ in the following way:

$$A_k(\omega) = \left\{ n \geqq 0 \;\middle|\; \frac{S}{2^{(k+1)/p}} < \| dX_n(\omega) \| \leqq \frac{S}{2^{k/p}} \right\}.$$

The following properties of these subsets are easily checked:

(i) If $k' \neq k''$ then $A_{k'}(\omega) \cap A_{k''}(\omega) = \phi$ and $\cup_{k\geqq 0} A_k(\omega) = \{n \geqq 0 \mid \| dX_n(\omega) \| \neq 0\}$ for all ω in Ω.

(ii) For each $k \geqq 0$, the cardinal of $A_k(\omega)$ noted $|A_k(\omega)|$ satisfies: $\forall \omega \in \Omega$, $|A_k(\omega)| < 2^{k+1}$.

(iii) For each $k \geqq 0$ and each $n \geqq 1$ the set $\{\omega \in \Omega \mid n \in A_k(\omega)\}$ is in $\mathcal{A}_{n-1}$.

We immediately deduce from (i) that:

$$\| X_N \|_2 \leqq \sum_{k\geqq 0} \left\| \sum_{n \in A_k(\cdot)} dX_n(\cdot) \right\|_2 . \tag{3.9}$$

For each $k \geqq 0$ we can define a sequence of stopping times $(T_\lambda^k)_{\lambda \geqq 0}$ by setting: $\forall \omega \in \Omega$

$T_0^k(\omega) = \inf\{n \in A_k(\omega)\}$ and for $\lambda = 1, 2, \cdots$

$T_\lambda^k(\omega) = \inf\{n \in A_k(\omega), n > T_{\lambda-1}^k(\omega)\}$ with the convention $\inf \phi = N + 1$.

By property (iii) these stopping times are "predictable", that is: $\forall\lambda \geqq 1$, $T^k_\lambda - 1$ is also a stopping time; therefore $\forall\lambda \geqq 1$

$$\mathbf{E}^{\mathscr{A}_{T^k_{\lambda-1}}}(dX_{T^k_\lambda(\cdot)}(\cdot)) = 0. \tag{3.10}$$

By the definition of the $(T^k_\lambda)_{\lambda\geqq 0}$ and by (ii):

$$A_k(\omega) \subset \bigcup_{0\leqq\lambda<2^{k+1}} \{T^k_\lambda(\omega)\};$$

therefore, since $dX_{N+1} = 0$:

$$\sum_{n\in A_k(\omega)} dX_n(\omega) = \sum_{0\leqq\lambda<2^{k+1}} dX_{T^k_\lambda(\omega)}(\omega).$$

Now if for each $\lambda \geqq 0$ we define $\tilde{X}^k_\lambda(\cdot)$ as

$$\sum_{0\leqq\mu\leqq\lambda} dX_{T^k_\mu(\cdot)}(\cdot),$$

then (3.10) ensures that $(\tilde{X}^k_\lambda)_{\lambda\geqq 0}$ is a martingale with respect to the sequence of σ-algebras $(\mathscr{A}_{T^k_\lambda})_{\lambda\geqq 0}$; applying (3.6) to this martingale, we obtain: $\forall k \geqq 0$

$$\Big\| \sum_{n\in A_k(\cdot)} dX_n(\cdot) \Big\|_2 = \| \tilde{X}^k_{2^{k+1}-1} \|_2 \leqq D2^{(k+1)/r} \sup_{0\leqq\lambda<2^{k+1}} \| d\tilde{X}^k_\lambda \|_\infty; \tag{3.11}$$

returning to the definition of $A_k(\cdot)$, we observe that for all $\lambda \geqq 0$:

$$\| d\tilde{X}^k_\lambda \|_\infty \leqq \frac{S}{2^{k/p}}. \tag{3.12}$$

Combining (3.12), (3.11) and (3.9), we obtain:

$$\| X_N \|_2 \leqq D2^{1/r} \Big(\sum_{k=0}^{\infty} 2^{k(1/r-1/p)} \Big) S.$$

This concludes the proof (since $1/r - 1/p < 0$) with

$$\alpha_p = D2^{1/r} \sum_{k\geqq 0} 2^{k(1/r-1/p)} < \infty.$$

SUBLEMMA 3.2. *Assume that the conclusion of Sublemma 3.1 is valid, then for all $p < r$ there exists a constant β_p such that all X-valued Walsh-Paley martingales $(X_m)_{m\geqq 0}$ satisfy*

$$\Big(\sup_{c>0} c^{p/2} \mathbf{P}\{\sup \| X_n \| > c\} \Big)^{2/p} \leqq \beta_p \Big(\sum_{n\geqq 0} \mathbf{E} \| dX_n \|^p \Big)^{1/p}. \tag{3.13}$$

PROOF. If $(X_m)_{m\geqq 0}$ is an X-valued martingale, then $(\| X_m \|^2)_{m\geqq 0}$ is a submartingale, therefore we have by Doob's inequality ([21], p. 69):

(3.14) $$\forall c > 0, c^2 \mathbf{P}(\sup \| X_m \| > c) \leqq \sup \mathbf{E} \| X_m \|^2.$$

Consider an X-valued Walsh-Paley martingale $(X_m)_{m \geqq 0}$ such that $\Sigma_{n \geqq 0} \mathbf{E} \| dX_n \|^p \leqq 1$, and set for $n = 0, 1, \cdots$, $V_n = \Sigma_{0 \leqq k \leqq n} \| dX_k \|^p$. Given $a > 0$, we can define a stopping time T by setting:

$$\forall \omega \in \Omega,\ T(\omega) = \inf\{n \geqq 0 \mid V_{n+1} > a^p\}.$$

We can write, for $c > 0$

(3.15) $$\mathbf{P}(\sup \| X_n \| > c) \leqq \mathbf{P}(T < \infty) + \mathbf{P}(T = \infty,\ \sup \| X_n \| > c) \leqq \mathbf{P}(T < \infty) + \mathbf{P}\{T > 0,\ \sup \| X_{n \wedge T} \| > c);$$

applying (3.14) and (3.8) to the martingale $(1_{\{T>0\}} X_{n \wedge T})_{n \geqq 0}$, we get:

$$\mathbf{P}(T > 0, \sup \| X_{n \wedge T} \| > c) \leqq \frac{1}{c^2} \sup_n \mathbf{E}(1_{\{T>0\}} \| X_{n \wedge T} \|^2)$$
$$\leqq (\alpha_p^2 / c^2) \| 1_{\{T>0\}} V_T^{1/p} \|_\infty^2.$$

The definition of T yields: $1_{\{T>0\}} V_T \leqq a^p$, therefore we get:

$$\mathbf{P}(T > 0, \sup \| X_{n \wedge T} \| > c) \leqq \alpha_p^2 (a/c)^2;$$

on the other hand,

$$\mathbf{P}(T < \infty) = \mathbf{P}\left\{\sup_n V_n > a^p\right\} \leqq 1/a^p.$$

With these estimates (3.15) implies:

$$\mathbf{P}(\sup \| X_n \| > c) \leqq 1/a^p + \alpha_p^2 (a/c)^2$$

for all $a, c > 0$. Choosing (for simplicity) $a = \sqrt{c}$, we get $\forall c \geqq 1$,

$$\mathbf{P}(\sup \| X_n \| > c) \leqq (1 + \alpha_p^2)(1/c^{p/2}),$$

which concludes the proof with $\beta_p = (1 + \alpha_p^2)^{2/p}$.

The argument of the above proof is classical, and appears in almost every book concerned with martingale inequalities in the scalar case ([12], [21]).

The proof of the next lemma is directly inspired from the methods in [5]. Its redaction has been simplified thanks to an observation of T. Figiel.

SUBLEMMA 3.3. *Assume that the conclusion of Sublemma 3.2 is valid, then for all $p < r$ there exists a constant γ_p such that all X-valued Walsh-Paley martingales $(X_n)_{n \geqq 0}$ satisfy:*

(3.16) $$\sup_{c>0} c^p \mathbf{P}\left\{\sup_n \| X_n \| > c\right\} \leqq (\gamma_p)^p \sum_{n \geqq 0} \mathbf{E} \| dX_n \|^p.$$

PROOF. Let us assume that $(X_m)_{m\geqq 0}$ is a "finite" Walsh-Paley martingale, so that $dX_m = 0$ for all $m \geqq N$ and that $\Sigma_{n\geqq 0}\mathbf{E}\| dX_n \|^p \leqq 1$. Let M be a "large" integer. We define the shift θ on $\Omega = \{-1, +1\}^{\mathbf{N}}$ by; $\theta(\varepsilon_1, \varepsilon_2, \cdots) = (\varepsilon_2, \varepsilon_3, \cdots)$, and we consider the Walsh-Paley martingale $(Z_m)_{m\geqq 0}$ uniquely defined by:

$$\forall m \geqq MN,\ Z_m = \left[X_{N-1} + \sum_{j=1}^{j=M-1} \varepsilon_{jN} X_{N-1} \circ \theta^{jN} \right] M^{-1/p}.$$

As is readily seen, $\Sigma_{m=0}^{\infty}\mathbf{E}\| dZ_m \|^p \leqq 1$. To lighten the notations, we denote the random variable $\sup_{n\geqq 0}\| X_n \|$ by Φ.

We observe that

$$\sup_{0\leqq j \leqq M-1} M^{-1/p}\Phi\circ\theta^{jN} \leqq 2 \sup_{n\geqq 0}\| Z_n \|.$$

We choose c_0 large enough so that $(2\beta_p/c_0)^{p/2} \leqq 1 - e^{-1}$, and we apply 3.13 to the martingale $(Z_m)_{m\geqq 0}$:

$$\mathbf{P}\left\{ \sup_{0\leqq j\leqq M-1} M^{-1/p}\,\Phi\circ\theta^{jN} > c_0 \right\} \leqq \mathbf{P}\left\{ \sup_{n\geqq 0}\| Z_n \| > c_0/2 \right\}$$

$$\leqq (2\beta_p/c_0)^{p/2} \leqq 1 - e^{-1}.$$

When $j = 0, \cdots, M-1$, the variables $\Phi\circ\theta^{jN}$ are independent and equidistributed, therefore:

$$e^{-1} \leqq \mathbf{P}\left\{ \sup_{0\leqq j\leqq M-1} M^{-1/p}\,\Phi\circ\theta^{jN} \leqq c_0 \right\} = (\mathbf{P}\{M^{-1/p}\,\Phi \leqq c_0\})^M,$$

as a consequence: $\mathbf{P}\{\Phi > c_0 M^{1/p}\} \leqq 1 - e^{-1/M} \leqq 1/M$, which we proved for an arbitrary integer M. From this last fact, (3.16) is easily drawn.

We can now conclude the proof of Lemma 3.1 by an argument similar to that of Sublemma 3.2:

Let p be given with $p < r$; we choose p_1 with $p < p_1 < r$. If $(X_m)_{m\geqq 0}$ is an X-valued Walsh-Paley martingale, we define for each $c > 0$ a stopping time T by:

$$\forall\omega\in\Omega,\ T(\omega) = \inf\left\{ n \geqq 0 \sum_{0\leqq j\leqq n+1} \| dX_j(\omega) \|^{p_1} > c^{p_1} \right\}.$$

We have:

$$\mathbf{P}\{\sup\| X_n \| > c\} \leqq \mathbf{P}\{T < \infty\} + \mathbf{P}\{T = \infty \sup\| X_{n\wedge T}\| > c\}$$

$$\leqq \mathbf{P}\left\{ \left(\sum_{n\geqq 0} \| dX_n \|^{p_1} \right)^{1/p_1} > c \right\} + \left(\frac{\gamma_{p_1}}{c} \right)^{p_1} \mathbf{E}\left(1_{\{T>0\}} \sum_{j=0}^{j=T} \| dX_j \|^{p_1} \right)$$

$$\leqq \mathbf{P}\left\{ \left(\sum_{n\geqq 0} \| dX_n \|^{p_1} \right)^{1/p_1} > c \right\} + \left(\frac{\gamma_{p_1}}{c} \right)^{p_1} \mathbf{E}\left(c^{p_1} \wedge \sum_{n\geqq 0} \| dX_n \|^{p_1} \right).$$

If we multiply the last line by $pc^{p-1}dc$ and integrate with respect to c, we get:

$$\mathbf{E}\sup\|X_n\|^p \leqq \mathbf{E}\Big(\sum_{n\geqq 0}\|dX_n\|^{p_1}\Big)^{p/p_1}\Big[1+(\gamma_{p_1})^{p_1}+(\gamma_{p_1})^{p_1}\frac{p}{p_1-p}\Big];$$

it remains to observe that:

$$\sum_{n\geqq 0}\|dX_n\|^{p_1} \leqq \Big(\sum_{n\geqq 0}\|dX_n\|^p\Big)^{p_1/p},$$

to conclude the proof of Lemma 3.1.

The final step can be proved alternatively by an argument of interpolation.

REMARK 3.3. Let α, p, q be given in $[1,\infty[$ and consider the following properties $\mathcal{S}_p(\alpha)$, $\mathcal{C}_q(\alpha)$ for a Banach space X:

$\mathcal{S}_p(\alpha)$: There exists a constant $s_p(\alpha)$ such that all X-valued martingales $(X_m)_{m\geqq 0}$ satisfy:

$$\mathbf{E}\sup_{n\geqq 0}\|X_n\|^\alpha \leqq s_p(\alpha)\mathbf{E}\Big\{\Big(\sum_{n\geqq 0}\|dX_n\|^p\Big)^{\alpha/p}\Big\}.$$

$\mathcal{C}_q(\alpha)$: There exists a constant $c_q(\alpha)$ such that all X-valued martingales $(X_m)_{m\geqq 0}$ satisfy

$$\mathbf{E}\Big\{\Big(\sum_{n\geqq 0}\|dX_n\|^q\Big)^{\alpha/q}\Big\} \leqq c_q(\alpha)\mathbf{E}\sup_{n\geqq 0}\|X_n\|^\alpha.$$

We have seen (Theorem 3.1 and Proposition 2.4) that X is p-smooth iff X has the property $\mathcal{S}_p(p)$ and that X is q-convex iff X has the property $\mathcal{C}_q(q)$. By reproducing the basic techniques of the proofs of the Burkholder-Davis-Gundy inequalities, it is possible to show that: (i) for each p in $]1,2]$, all the properties $\mathcal{S}_p(\alpha)$ are equivalent when α runs in $[1,\infty[$. (ii) for each q in $[2,\infty[$, all the properties $\mathcal{S}_q(\alpha)$ are equivalent when α runs in $[1,\infty[$.

We give only indications on the proof: we first notice that the Burgess-Davis decomposition (see [12], p. 91) is obviously true for Banach space valued martingales with the same proof. If $1 \leqq \alpha < \beta$, the proof that $\mathcal{S}_p(\beta) \Rightarrow \mathcal{S}_p(\alpha)$ and $\mathcal{C}_q(\beta) \Rightarrow \mathcal{C}_q(\alpha)$ can be achieved by reproducing, for instance, the method of [21], page 181–182. Taking into account the obvious equivalence of "X has $\mathcal{S}_p(\alpha)$" and "X^* has $\mathcal{C}_{p'}(\alpha')$" when $\alpha \in]1,\infty[$ and $1/\alpha + 1/\alpha' = 1$, $1/p + 1/p' = 1$, we observe that it remains only to prove that $\mathcal{S}_p(1)$ implies $\mathcal{S}_p(\alpha)$ for some $\alpha > 1$ and that $\mathcal{C}_q(1)$ implies $\mathcal{C}_q(\alpha)$ for some $\alpha > 1$. These last implications can be proved easily by using the results of [5], as we already did in the proof of Sublemma 3.3.

The above remark was independently observed by P. Assouad.

4. Remarks and problems

I do not know if Lemma 3.1 can be proved assuming only that (3.6) is satisfied by all the X-valued *Walsh-Paley* martingales. The following question is related to this problem:

QUESTION 1. Let q be given in $]2,\infty[$. Assume that for any equivalent norm on a Banach space X the corresponding modulus of convexity satisfies $\delta(\varepsilon)/\varepsilon^q \not\to \infty$ when ε tends to zero. Is it true that there exists for all n and $d < 1$ an X-valued Walsh-Paley martingale $(X_m)_{m \geqq 0}$ with the property that: $\| X_n \|_\infty \leqq 1$ and $\forall \omega \in \Omega$, $\forall k = 1,2,\cdots,n$, $\| dX_k(\omega) \| \geqq d/n^{1/q}$?

A positive answer to Question 1 would give a natural extension of Theorem 1.1.b.

B. Beauzamy has defined in [2] the notion of uniform convexity of an operator. If u is an operator from Y to X, the modulus of convexity of u is defined as:

$$\delta_u(\varepsilon) = \inf\left\{ 1 - \left\| \frac{x+y}{2} \right\| \;\middle|\; x, y \in Y,\ \| x \| \leqq 1,\ \| y \| \leqq 1,\ \| u(x-y) \| \geqq \varepsilon \right\}.$$

Similarly, one can define the modulus of smoothness of u as:

$$\rho_u(t) = \sup\left\{ \frac{\| x + tu(y) \| + \| x - tu(y) \|}{2} - 1 \;\middle|\; x \in X,\ y \in Y,\ \| x \| = \| y \| = 1 \right\}.$$

It is clear that some of our results immediately extend to the case of operators with mere notational changes (at least Theorem 3.1 and Lemma 3.1). But, as pointed in [2], the analogue of Theorem 3.2 for operators is false.

A Banach space X is called of type p if for each sequence (x_n) of elements of X such that $\Sigma \| x_n \|^p < \infty$, the Rademacher series $\Sigma r_n(t) x_n$ is convergent for almost every t in $[0,1]$ (we denote (r_n) the Rademacher functions on the Lebesgue interval).

A counterexample due to James ([15]) shows that there exists a Banach space which is of type p for some $p > 1$ but which is p-smooth for no $p > 1$. In view of this counterexample, the following question of H. P. Rosenthal is of particular interest:

QUESTION 2. Is every space of type 2 super-reflexive?

We wish to give a (very) partial result on this question: Let $(g_n)_{n\geqq 1}$ be a sequence of orthonormal Gaussian variables with mean zero on the Lebesgue interval, and let X be a Banach space. We define for each $n = 1,2,\cdots$ the numbers a_n^X and b_n^X as the smaller positive constants respectively a and b such that

$$\frac{1}{b}\left(\sum_{i=1}^{n}\|x_i\|^2\right)^{1/2} \leqq \left\|\sum_{i=1}^{n} g_i x_i\right\|_2 \leqq a\left(\sum_{i=1}^{n}\|x_i\|^2\right)^{1/2}$$

for all n-tuples $(x_1,\cdots,x_n)$ of elements of X.

S. Kwapień has proved in [16] that $\sup_n a_n^X b_n^X < \infty$ if and only if X is isomorphic to a Hilbert space. Using the same idea as his, we prove:

Proposition. *If $a_n^X b_n^X/\text{Log}\, n$ tends to zero when $n\to\infty$, then X is super-reflexive.*

Proof. Let (u_{ij}) be the $n\times n$ matrix of an isometry on l_2^n. By the rotational invariance of the canonical Gaussian measure on l_2^n, we have, for all n-tuples (x_i) of elements of X:

$$\left(\sum_{i=1}^{n}\left\|\sum_{j=1}^{n} u_{ij}x_j\right\|^2\right)^{1/2} \leqq b_n^X\left\|\sum_{i=1}^{n} g_i x_i\right\|_2 \leqq a_n^X b_n^X\left(\sum_{i=1}^{i=n}\|x_i\|^2\right)^{1/2}.$$

Since matrices such as $u_{i,j}$ correspond to the extremal points of the unit ball of $B(l_2^n)$, it is clear, by an argument of convexity, that the above inequality remains valid if $u_{i,j}$ defines an operator of norm less than 1 on l_2^n. The Hilbert matrix (h_{ij}^n) of order n is defined by:

$$\forall i,j\leqq n,\quad h_{ij}^n = \begin{cases} \dfrac{1}{n+1-i-j} & \text{if } i+j\neq n+1 \\ 0 & \text{otherwise}\end{cases}$$

It is known (see [25]) that when n goes to infinity the norm of (h_{ij}^n) as an operator on l_2^n is bounded by some constant H.

Assume that X is not super-reflexive, then ([13]) for all integer n there exists $x_1,\cdots,x_n$ in the unit ball of X such that: $\forall(\alpha_i)\in\mathbf{R}^n$

$$\frac{1}{2}\sup_{k\leqq n}\left|\sum_{i=1}^{i=k}\alpha_i\right| \leqq \left\|\sum_{i=1}^{n}\alpha_i x_i\right\|;$$

therefore we must have: $\forall(\alpha_i)\in\mathbf{R}^n$

$$\frac{1}{2}\left(\sum_{i=1}^{n}\sup_{k\leqq n}\left|\sum_{j=1}^{k} h_{ij}^n\alpha_j\right|^2\right)^{1/2} \leqq H a_n^X b_n^X\left(\sum_{i=1}^{i=n}|\alpha_i|^2\right)^{1/2}.$$

If we denote by φ_n the left-hand side of the preceding inequality for $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 1/\sqrt{n}$, an elementary computation shows that $\varphi_n/\mathrm{Log}\, n$ does not tend to zero when $n \to \infty$; therefore $a_n^X b_n^X/\mathrm{Log}\, n$ does not either, which proves the proposition.

As noticed by T. Figiel, an improvement of the methods of [10] yields that if a Banach space X with local unconditional structure is of type p, $p \leqq 2$, then X is p-smooth; the example in [15] suggests

QUESTION 3. If $p \in]1,2]$, does there exist a super-reflexive Banach space of type p which is not p-smooth?

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